Effect of a Cylindrical Boundary on a Fixed Rigid Sphere in a Moving Viscous Fluid

A. M. FAYON and JOHN HAPPEL
New York University, New York, New York

In this investigation of the effect of a cylindrical boundary on a spherical particle suspended in a moving viscous fluid the pressure drop due to the motion of fluid past a stationary sphere and the drag on the sphere are independently measured, and the sphere- to-cylinder-diameter ratios varying from 0.1250 to 0.3125 are studied at different positions of eccentricity. The investigation shows that in the range of Reynolds numbers from 0.1 to 40.0, based on the approach velocity to the sphere and the diameter of the sphere, the pressure drop due to the presence of a sphere and the drag on it can be represented by a semiempirical relationship containing two terms. One term, owing to the effect of the cylindrical boundary, is derived theoretically from the creeping-motion equations which neglect inertial effects. The second term, owing to the inertial effects, is established from data on the drag coefficient of a sphere in an unbounded medium.

The effect of the boundary on the resistance to flow through an assemblage of particles in a moving fluid is important in problems involving such phenomena as fluidization, sedimentation, and flow through fixed beds. To obtain an insight into the fundamental hydrodynamics involved in problems of this type, the motion of a sphere in a cylinder was selected as a mathematical model.

When motion occurs at low Reynolds numbers, a suitable theoretical treatment is possible by omitting the inertial terms from the Navier-Stokes equations of motion to obtain the so-called "creeping-motion" equations. On the basis of these equations theoretical studies (*4, 7*) indicate that for the case where a sphere settles axially in a vertical cylinder containing a viscous fluid which is stationary, the drag is

$$\frac{W}{6\pi\mu u\, a} = \frac{1}{1-2.105\left(\frac{a}{R}\right)+2.087\left(\frac{a}{R}\right)^3} \quad (1)$$

An extension of this treatment (*6, 11*) gives the following expression for the drag on a stationary sphere at the axis of a cylindrical tube through which fluid is moving:

$$\frac{W}{6\pi\mu u_o\, a} = \frac{1-\frac{2}{3}\left(\frac{a}{R}\right)^2}{1-2.105\left(\frac{a}{R}\right)+2.087\left(\frac{a}{R}\right)^3} \quad (2)$$

The case for sedimentation has been checked experimentally, but that for the drag on a stationary sphere has not. Haberman (5) has developed more exact equations for the above two cases, especially applicable to high sphere-to-cylinder-diameter ratios.

Faxen (*4*) on the basis of Oseen's (*9*) equation, which partially takes inertial terms in the Navier-Stokes equations into consideration, derived the following expression for a sphere settling axially:

$$\frac{W}{6\pi\mu u a} = \frac{1}{1-\frac{3}{16}N_{Re_o}-\frac{d}{D}\left[f\left(N_{Re_o}\frac{D}{d}\right)\right]+2.09\left(\frac{d}{D}\right)^3} \quad (3)$$

Faxen calculated $f(N_{Re_o}D/d)$ as equal to 2.105 for N_{Re_o} equal to zero. He obtained values for $f(N_{Re_o}D/d)$ which decreased rapidly as N_{Re_o} increased.

McNown *et al.* (*8*) conducted an extensive experimental study for the case of a sphere sedimenting along the axis of a cylinder. Reynolds numbers up to 70 and d/D from 0.2 to 0.8 were included.

Brenner and Happel (*1*) developed solutions of the creeping-motion equations for the case of a sphere located eccentrically in a cylinder; they gave the following theoretical equation for a stationary sphere in a moving fluid:

$$\frac{W}{6\pi\mu u_o a} = (1-\beta^2)\left[1-F(\beta)\frac{a}{R}+\left\{F^2(\beta)-\frac{2}{3(1-\beta^2)}\right\}\left(\frac{a}{R}\right)^2\ldots\right] \quad (4)$$

where $-F(\beta) = 2.105 - 0.6977\,\beta^2$

and $\beta = \frac{b}{R}$

Equation (4) may be written as follows to preserve analogy with Equations (1) and (2):

$$\frac{W}{6\pi\mu u_o a(1-\beta^2)} = \frac{1-\frac{2}{3(1-\beta^2)}\left(\frac{a}{R}\right)^2}{1-F(\beta)\frac{a}{R}} \quad (4a)$$

or for the case of a sedimenting sphere

$$\frac{W}{6\pi\mu u a} = \frac{1}{1-F(\beta)\left(\frac{a}{R}\right)} \quad (4b)$$

For the case of sedimentation of an eccentric sphere the experimental data of Craig (*2*) show that for values of $\beta \leq 0.5$, $-F(\beta)$ will be close to 2.105, as indicated by the expression above. It was also not possible in this work to establish the validity of the constant 0.6977, which has only a small effect on $F(\beta)$ in the range where it is applicable (that is near the axis of the cylinder).

The purpose of the present investigation was to study the effect of higher Reynolds numbers on the drag of a stationary sphere at various eccentricities. This technique is superior to sedimentation, because it is possible to support the sphere in a fixed position. Otherwise it would tend, in either case, to move toward the center of the tube, owing to inertial effects.

DESCRIPTION OF APPARATUS

The apparatus for measuring the pressure drop and drag on a single sphere in a moving fluid consisted of a glass tube 4 in. in diameter and 1 ft. long, flanged with gaskets at both ends to sections of 4-in. standard brass pipe. The upper section was 1½ ft. long and the lower section 4½ ft. long, the total length of the column being 7 ft. The circulating system was started with a 50 gal./min. capacity gear pump, which was provided with a bypass and driven by a 5-hp. motor.

The circulating fluid, which consisted of a 98% glycerol solution in water, entered the column through one of two rotameters, having a capacity of 0.3 to 3 and 3 to 38 gal./min. respectively. From the rotameter the liquid was introduced to the bottom of the column, upward through the column, and from the top back to a recirculating drum, which was provided with cooling coils. All flow lines in and out of the column, leading to the various pieces of the experimental setup, were 1.5-in. standard brass pipe.

Pressure taps were mounted 18.2 in. apart on the upper and lower side of the 4-in. brass pipe, near the flanges which held the 4-in. glass tube. The pressure taps were connected to a differential hook gauge which consisted of two glass tubes 4 in. long and 2.5 in. in diameter, flanged on both ends through suitable gaskets to prevent leakage and connected to each other from the upper flanges by copper tubing. Each glass tube had an opening at the bottom flange to connect to the leads from the pressure taps and contained two immiscible liquids at different densities; the heavier liquid was glycerol, the circulating liquid, and the lighter was dye-colored benzene. A micrometer was mounted at the bottom of each container, with the spindle extending vertically upward into the liquid. The end of the spindle was sharpened to a point which could then be raised and lowered to locate the position of the interface in both glass tubes with an accuracy of ± 0.002 in. The gauge was provided with a suitable venting device.

Thermometers were inserted at the inlet and exit side of the column, at the exit of the rotameters, and adjacent to the hook gauge.

In measuring the pressure drop due to a sphere, the sphere in question was mounted on the end of a 0.0625-in. wire connected to a rod having a diameter of 0.22 in. A 0.25-in. thick brass sliding plate was held in position at the upper end of the column by a brass ring having six uniformly spaced thumb screws. This sliding plate was provided with a vent valve and a packing gland, through which the 0.22-in. rod was inserted. This rod could be raised and lowered to introduce the sphere and part of the 0.0625-in. wire into the pressure-tap zone. The position of the sphere was at equal distances from either pressure tap. The wire and sphere could also be removed from the pressure-tap zone. The sliding plate could be moved horizontally so as to place the sphere in any desired position of eccentricity.

To measure the drag on a sphere the upper end of the column was modified as follows. The sliding plate was replaced with a plate of the same thickness, through which a 1.25-in. hole was drilled. An analytical balance was mounted on a platform above the column. The distance between the tray of the balance and the upper end of the column was 18 in. A hook was attached to this tray and held a 0.01-in. wire, which was connected to the sphere and also extended to the glass section of the column so that the sphere might be observed at a point equidistant from both ends of the glass section of the column.

EXPERIMENTAL PROCEDURE

Since the data on drag are more accurate than those on pressure drop, only the former are treated in this paper. The procedure for obtaining the drag measurements follows.

A run was begun by taring the sphere and the wire on the balance with no fluid flowing through the column. The fluid was then circulated at a desired flow rate until no change in rotameter setting and fluid temperature could be observed. The sphere and wire were then reweighed with the fluid flowing through the column. The difference between this reading and the original reading was the drag due on the sphere and the wire. To measure the drag on the wire alone, the experiment was repeated under the same conditions at an additional wire length. The difference of the two readings at the two different wire lengths was due to the drag on the wire alone. This quantity was subtracted from the original reading, and thus the drag on the sphere alone was obtained.

CORRELATION AND DISCUSSION OF THE DATA*

The drag on the sphere was measured at Reynolds numbers ranging from 0.1 to 20 on the basis of the average linear velocity and sphere diameter. Spheres of diameter sizes of 1.250, 0.998, 0.750, and 0.5 in. were used. The sphere- to cylinder-diameter ratio for each of these spheres was 0.3125, 0.2495, 0.1875, and 0.1250. In addition to placing the sphere at the cylinder axis, two off-cylinder axis positions of 0.5 and 1.0 in. were studied. (Owing to the great quantity of data, only the experiments involving the 1.250-in. sphere are presented here.)

The experimental data were correlated in such a fashion as to separate the wall effect from the inertial effects. The data were fitted by the following semiempirical equation:

$$\frac{W}{6\pi\mu u_o a(1-\beta^2)} = \left[\frac{1-\frac{2}{3(1-\beta^2)}\left(\frac{a}{R}\right)^2}{1-K\frac{a}{R}+2.087\left(\frac{a}{R}\right)^3} + \left(\frac{C_A}{C_S}-1\right)\right] \quad (5)$$

where K is equal to 2.105 according to the theoretical developments noted previously. The term $[(C_A/C_S)-1]$ appearing in Equation (5) is due to inertial effects and is dependent on the Reynolds number only. This term is obtained by simply considering the actual drag on a sphere at an infinite distance from the cylinder wall and comparing it with the drag according to Stokes's law, which also applies to an unbounded system. C_A is the actual drag coefficient for a sphere in an unbounded system taken from experimental data *(10a)* and C_S the drag coefficient according to Stokes's law. C_A and C_S are evaluated at the Reynolds number corresponding to the approach velocity to the sphere. The term $[(C_A/C_S)-1]$ therefore represents the fractional deviation of the actual drag from the drag according to Stokes's law.

Multiplying the fluid velocity at the

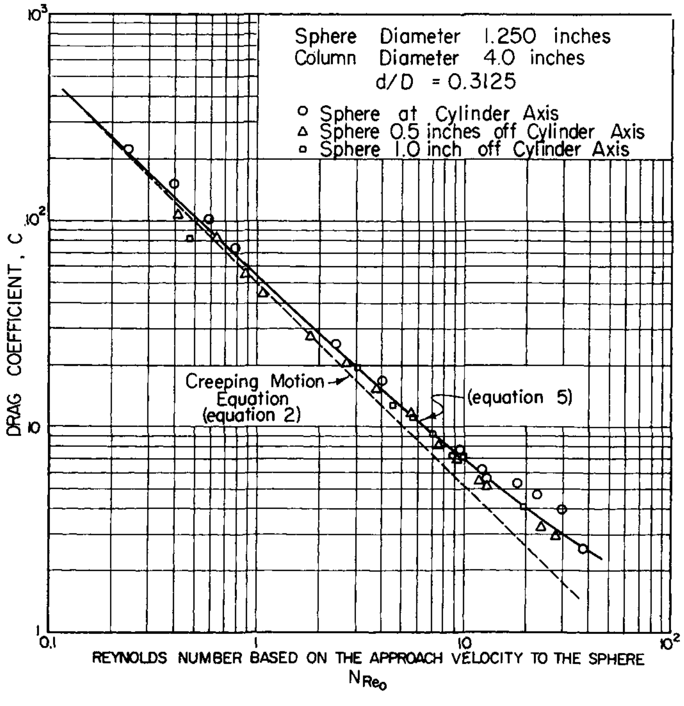

Fig. 1. Drag on a single stationary sphere at three different eccentricities.

* Part of the results are summarized in Figure 1. Complete detailed data appear in reference 3.

cylinder axis by $(1-\beta^2)$, as it appears in the denominator in the left-hand side of Equation (5), one obtains the approach velocity to the sphere.

The first term in the right-hand side of Equation (5) represents the effect of the cylindical boundary and is the same as derived from the creeping-motion equations. As calculated from this equation, the coefficient of (a/R) designated here as K is equal to 2.105. A test of the hypothesis that the wall effect and inertial effects are additive as represented by Equation (5), where the wall effect is independent of the Reynolds number, would be to determine whether or not K is constant over the whole range of Reynolds numbers investigated.

The constant K was determined from Equation (5) for each of the 118 runs performed; the average value was 1.959. A statistical analysis showed the standard deviation to be 0.383 from the average value.

The values of K were very consistent for the runs involving the spheres of 1.250- and 0.998-in. diameter. Thirty-five runs were performed with the 1.250-in. sphere in the Reynolds number range of 0.2 to 37.0, based on the approach velocity to the sphere and the diameter of the sphere. For these runs the average value of K was 2.017 and the standard deviation equal to 0.180 from the average value.

The average value of K obtained from all the experimental data, equal to 1.959, differs by 6.8% from the value of 2.105 derived from the creeping-motion equation. Owing to the high standard deviation, there is no way of telling whether the difference between these two values is significant. The value of 2.105 for K, the coefficient of a/R, was adopted in the semiempirical Equation (5).

Equation (5) reduces to Equation (2) for the drag on a sphere located at the cylinder axis at Reynolds numbers smaller than 0.3. For this case $\beta = 0$ and $[(C_A/C_S)-1] = 0$. On the other hand, when the sphere is at an infinite distance from the wall and at Reynolds numbers smaller than 0.3, $\beta = 0$, $a/R = 0$, and $[(C_A/C_S)-1] = 0$; hence Equation (5) reduces to Stokes's law. If however for the latter case the Reynolds number is higher than 0.3, Equation (5) reduces to

$$\frac{W_A}{6\pi\mu u_o a} = \frac{C_A}{C_S} \tag{6}$$

In Figure 1 the drag coefficient calculated from the experimental data for the sphere with diameter of 1.250 in., which the authors consider the most accurate, is plotted as a function of the Reynolds number for all three sphere positions. The drag coefficient and the Reynolds number are based on the approach velocity to the sphere. The experimental data are compared with the creeping-motion equation [Equation (2)] and with the semiempirical Equation (5). At increasing Reynolds numbers, where inertial effects become appreciable, the experimental results, as had been expected, are higher than the results predicted from the creeping-motion equations. The semiempirical Equation (5) checks the experimental data reasonably well. In other words, the deviation of the experimental data from the creeping-motion equations at higher Reynolds numbers can be accounted for by simply adding to these equations the term $[(C_A/C_S - 1)]$, which accounts for the inertial effects.

The method developed here may be adapted for comparison with the data for a sedimenting sphere along the axis of a cylinder of McNown *et al.* (8). In other words, one may assume that the wall effect, as derived by the creeping-motion equations, and inertial effects are additive. Equation (5) can be slightly modified for this purpose to

$$\frac{W}{6\pi\mu u\, a} = \left[\frac{1}{1-2.105\dfrac{a}{R}+2.087\left(\dfrac{a}{R}\right)^3} + \left(\frac{C_A}{C_S}-1\right)\right] \tag{7}$$

where the term $1/[1-2.105\,(a/R) + 2.087\,(a/R)^3]$ is the effect of the cylindrical boundary for a sphere sedimenting in a fluid.

In Figure 2 the drag coefficient obtained from Equation (7) is compared with the experimental results for sedimenting spheres of McNown *et al.* (8), who plotted the drag coefficient as a function of the Reynolds number at different d/D ratios. Their experimental points appearing in Figure 2 were obtained from this plot. Equation (7) is in fair agreement with McNown's data, which justifies the hypothesis that the wall effect and the inertial effects are additive whether or not the sphere is stationary in a moving fluid or sedimenting in a fluid. The drag coefficient obtained from Faxen's theory [Equation (3)] is also plotted in Figure 2. Up to Reynolds numbers of approximately 0.4 his theory is in agreement with experimental data. At higher Reynolds numbers, however, it deviates very sharply from the experimental data.

CONCLUSIONS

The study shows that over a range of Reynolds numbers of 0.1 to 40.0 based on the approach velocity to the sphere and the diameter of the sphere, the drag on a sphere surrounded by a cylindrical boundary, whether or not the sphere is stationary in a moving fluid or sedimenting in a fluid, can be expressed by a semiempirical equation composed of two additive terms. One term is due to the effect of the cylindrical boundary and the other due to inertial effects. Furthermore it can be

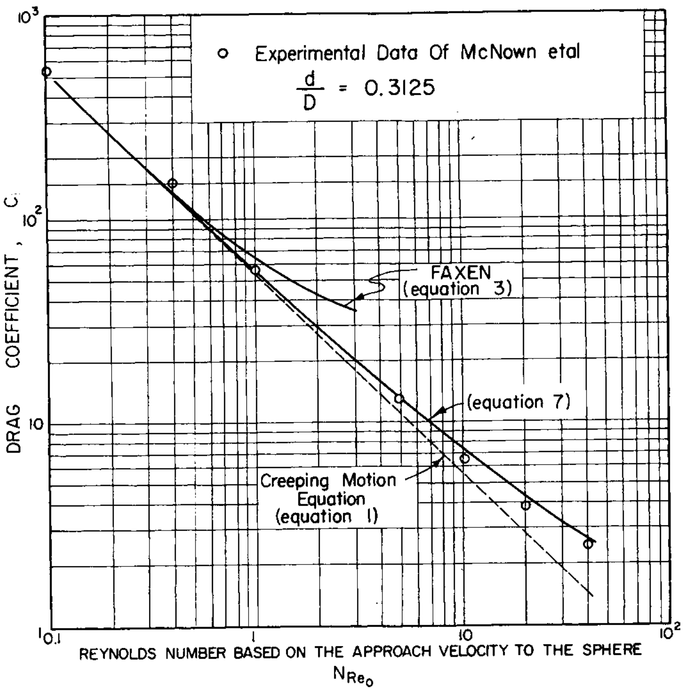

Fig 2. Drag on a single sedimenting sphere placed axially in a cylinder.

concluded that when the data are expressed as in Equation (5) or (7), the wall effect is a function of the (a/R) ratio only. Up to sphere- to-cylinder-diameter ratios of 0.3125 and eccentricities of 50% of the cylinder radius the effect of the cylindrical boundary is not affected by an increase of the Reynolds number and by various positions of eccentricity. It is the same as derived by the creeping-motion equations.

The present work does not agree with the theory of Faxen, who uses the Oseen approximation (9), where the value of $L|\delta L|$ in Equation (3) decreases with an increase of the Reynolds number.

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NOTATION

a = radius of sphere
b = cylinder axis to center of sphere distance
C = actual drag coefficient of a sphere in a bounded medium $= W/[\rho u_o^2/2)\pi a^2]$
C_A = actual drag coefficient of a sphere in an unbounded medium
C_S = drag coefficient according to Stokes's law equal to $24/N_{Re_o}$
$[(C_A/C_S)-1]$ = dimensionless group in Equations (5) and (7), indicating fractional deviation of actual drag from drag calculated by Stokes's law
d = diameter of sphere
D = diameter of cylinder
$F(\beta)$ = function in Equations (4), (4a), and (4b) defined by $(2.105 - 0.6977\ \beta^2)$
K = constant in Equation (5)
N_{Re_o} = Reynolds number based on approach velocity to the sphere $= [(d\,u_o\,\rho/\mu)](1-\beta^2)$
R = radius of cylinder
u_o = fluid velocity at cylinder axis
u = velocity of sphere
W = drag force on the sphere in a bounded medium
W_A = drag force on a sphere in an unbounded medium
β = dimensionless number equal to b/R
μ = viscosity of fluid
ρ = density of fluid

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Centrifugal Filtration Through Beds of Small Spheres

J. B. BINGEMAN and JESSE COATES
Louisiana State University, Baton Rouge, Louisiana

In 1856 Darcy observed that water flowed through a sand bed at a rate proportional to the area of the bed and to the difference between the upstream and dowstream hydraulic heads and inversely proportional to the bed thickness. Since that time many investigators have accumulated masses of data in support of numerous theoretical and empirical relationships for correlating and predicting pressure drop, flow rate, and fluid and bed properties. Naturally certain areas of research have received less attention than others.

Among the areas receiving least attention are flow through beds composed of particles of identical shape but mixed size, flow through unconsolidated beds composed of small particles, and flow under the influence of a centrifugal driving force. Accordingly, the flow of liquids through unconsolidated beds of small spherical particles of mixed size was investigated in two laboratory centrifugal filters. Experimental conditions were such that the flow was laminar and the cakes were incompressible. While there was some difficulty in obtaining reproducibility of cakes, the data for any particular cake were correlated satisfactorily by the Darcy equation adapted for centrifugal filtration.

THEORY

A mathematical expression for laminar flow of an incompressible fluid in a centrifugal filter may be derived from the classical Darcy equation, which is written

$$\frac{\Delta P}{L} = \frac{u\mu}{\alpha g_c} \tag{1}$$

Ruth (6) used an equivalent expression:

$$\frac{\Delta P}{L} = \frac{u\mu\alpha^1\rho_o}{g_c} \tag{2}$$

It is evident from Equations (1) and (2) that

$$\alpha^1\rho_o = \frac{1}{\alpha} \tag{3}$$

The Darcy equation may be adapted to centrifugal filtration as follows: For a centrifugal filter

$$\Delta P = \frac{a\rho_f H}{g_c} \tag{4}$$

where

$$a = \omega^2 r_{avg} = (2\pi N)^2 r_{avg} \tag{5}$$

and

$$H = \frac{\text{volume}}{\text{area}} = \frac{r_o^2 - r_f^2}{2r_{avg}} \tag{6}$$

Substitution of Equations (5) and (6) into Equation (4) yields

$$\Delta P = \frac{(2\pi N)^2 \rho_f (r_o^2 - r_f^2)}{2g_c} \tag{7}$$

Substituting Equation (7) into Equation (1) and noting that $u = \dfrac{Q_s}{A_c}$ leads to

$$\frac{Q_s}{A_c} = \frac{(2\pi N)^2 \alpha \rho_f (r_o^2 - r_f^2)}{2\mu L} \tag{8}$$

Taking A_c as

$$A_c = 2\pi r_{lm} h \tag{9}$$

and substituting in Equation (8) leads finally to the expression

$$Q_s = \frac{4\pi^3 N^2 \rho_f h \alpha (r_o^2 - r_f^2)}{\mu \ln \dfrac{r_o}{r_c}} \tag{10}$$

Equation (10) is the theoretical expression for the rate of flow of filtrate Q_s, of density ρ_f, and of viscosity through a bed of solids of permeability α in a centrifugal filter in which the liquid level is at r_f and the cake level is at r_c. The permeability has a constant value for an incompressible cake but changes with pressure when the cake is compressible.

The resistance of the filter medium to flow of the filtrate is not included in Equation (10). It may be taken into

J. B. Bingeman is with Ethyl Corporation, Baton Rouge, Louisiana.